Linear Operators Preserving the (p, q) Numerical Radius

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Dedicated to Professor Marvin Marcus on the occasion of his retirement

Submitted by Robert Crone

ABSTRACT

Let $\mathcal{M}$ be $\mathcal{H}_n$ or $\mathbb{C}_{n\times n}$, the real linear space of all $n \times n$ hermitian matrices and the complex linear space of all $n \times n$ complex matrices, respectively. For $1 \leqslant p \leqslant q \leqslant n$, the (p, q) numerical radius of $A \in \mathcal{M}$ is defined and denoted by

$$r_{p,q}(A) = \{|E_p(X^*AX)| : X \in \mathbb{D}_{n\times q},\ X^*X = I_q\},$$

where $E_p(Y)$ denotes the pth elementary symmetric function on the eigenvalues of Y. We characterize the linear operators ϕ on $\mathcal{M}$ that satisfy $r_{p,q}(\phi(A)) = r_{p,q}(A)$ for all $A \in \mathcal{M}$.

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1. INTRODUCTION

Let $\mathcal{M}$ be $\mathcal{H}_n$ or $\mathbb{C}_{n\times n}$, the real linear space of all $n \times n$ hermitian matrices and the complex linear space of all $n \times n$ complex matrices, respectively. For $1 \leqslant p \leqslant q \leqslant n$, the (p, q) *numerical range* of $A \in \mathcal{M}$ is defined and denoted by

$$W_{p,q}(A) = \{E_p(X^*AX) : X \in \mathbb{C}_{n\times q},\ X^*X = I_q\},$$

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where $E_p(Y)$ denotes the pth elementary symmetric function on the eigenvalues of Y, and the (p, q) *numerical radius* of $A \in \mathcal{M}$ is defined and denoted by

$$r_{p,q}(A) = \max\{|z| : z \in W_{p,q}(A)\}.$$

Let $Q_{q,n}$ denote the set of all strictly increasing sequences of length q whose terms belong to the set $\{1, \dots, n\}$. For $\omega \in Q_{q,n}$ and $Y \in \mathbb{C}_{n\times n}$ let $Y[\omega]$ be the $q \times q$ principal submatrix of Y lying in the rows and columns indexed by $\omega(1), \dots, \omega(q)$. One easily verifies that

$$r_{p,q}(A) = \max\{|E_p(UAU^*[\omega])| : \omega \in Q_{q,n}, UU^* = I_n\}.$$

Since $E_p(Y) = \operatorname{tr} C_p(Y)$, where $C_p(Y)$ denotes the pth compound of Y (see e.g. [13] for the definition and properties of compound matrices), one may replace $E_p(\cdot)$ by $\operatorname{tr} C_p(\cdot)$ in the above definitions. The concept of (p, q) numerical radius can also be considered in $\wedge^q(\mathbb{C}^n)$, the qth Grassmann space over $\mathbb{C}^n$. In fact, if $D_p^q(A)$ is the pth derivation of A on $\wedge^q(\mathbb{C}^n)$, i.e., the linear operator determined by the formula

$$C_q(I + tA) = \sum_{r=0}^{q} t^r D_r^q(A),$$

then (e.g., see [17])

$$r_{p,q}(A) = \max\{|\langle D_p^q(A)x^\wedge, x^\wedge\rangle| :$$

$$x^\wedge \text{ is a decomposable unit vector in } \wedge^q(\mathbb{C}^n)\}.$$

The definitions of $W_{p,q}(A)$ and $r_{p,q}(A)$ specialize to the following well-known concepts:

(a) If $p = 1$, we get the *q-numerical range* $W_q(A)$ and the *q-numerical radius* $r_q(A)$. If q also equals 1, then we get the *classical numerical range* $W(A)$ and the *classical numerical radius* $r(A)$.

(b) If $p = q$, we get the *qth decomposable numerical range* $W_q^\wedge(A)$ and the *qth decomposable numerical radius* $r_q^\wedge(A)$.

(c) If $q = n$, then we get the set $\{E_p(A)\}$ and the value $|E_p(A)|$. In particular, we get

(c.i) $\{\operatorname{tr} A\}$ and $|\operatorname{tr} A|$ if $p = 1$, and
(c.ii) $\{\det A\}$ and $|\det A|$ if $p = n$.

The study of (p, q) numerical ranges and (p, q) numerical radii has attracted the attention of many authors in the last few decades, and it has been shown that these concepts are very useful for studying matrices and linear operators (for example, see [2, 4, 12, 14] and their references). Very recently, the present author and his collaborators have discovered (see [3]) that the (p, q) numerical range is related to the study of a q-particle system with p-body interaction in quantum physics. This makes the subject more interesting. While many results have been obtained, there are still many open problems on the subject that deserve further research. In particular, there has been a great deal of interest in determining the structure of those linear operators satisfying

$$W_{p,q}(\phi(A)) = W_{p,q}(A) \qquad \text{for all} \quad A \in \mathscr{M}, \tag{1}$$

or

$$r_{p,q}(\phi(A)) = r_{p,q}(A) \qquad \text{for all} \quad A \in \mathscr{M}. \tag{2}$$

A linear operator on $\mathscr{M}$ satisfying (1) [respectively, (2)] is called a (p, q) *numerical range preserver* [respectively, (p, q) *numerical radius preserver*]. The structure of (p, q) numerical range preservers has been completely determined (for example, see [9] and its references). However, only the following special cases have been treated for the corresponding problem concerning (p, q) numerical radius preservers (see [7, 8, 10, 11]).

(a) Suppose $p = 1$. A linear operator ϕ on $\mathscr{M}$ satisfies (2) if and only if there exist a unitary matrix U and $\mu \in \mathbb{F}$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ according as $\mathscr{M} = \mathscr{H}_n$ or $\mathbb{C}_{n \times n}$, with $|\mu| = 1$ such that one of the following holds.

(i) One has

$$\phi(A) = \mu UAU^* \qquad \text{for all} \quad A \in \mathscr{M}. \tag{3}$$

(ii) One has

$$\phi(A) = \mu UA^tU^* \qquad \text{for all} \quad A \in \mathscr{M}. \tag{4}$$

(iii) $q = n/2$ and

$$\phi(A) = \mu\left(\frac{2}{n}(\operatorname{tr} A)I - UAU^*\right) \qquad \text{for all} \quad A \in \mathscr{M}. \tag{5}$$

(iv) $q = n/2$ and

$$\phi(A) = \mu\left(\frac{2}{n}(\operatorname{tr} A)I - UA^tU^*\right) \qquad \text{for all} \quad A \in \mathscr{M}. \tag{6}$$

(b) Suppose $p = q < n$. A linear operator ϕ on $\mathscr{M}$ satisfies (2) if and only if there exist a unitary matrix U and $\mu \in \mathbb{F}$ with $|\mu| = 1$ such that (3) or (4) holds.

(c) Suppose $p = q = n$. A linear operator ϕ on $\mathscr{M}$ satisfies (2) if and only if

$$\phi(A) = MAN \qquad \text{for all} \quad A \in \mathscr{M} \tag{7}$$

or

$$\phi(A) = MA^tN \qquad \text{for all} \quad A \in \mathscr{M} \tag{8}$$

for some $M, N \in \mathbb{C}_{n\times n}$ with $|\det MN| = 1$, and if $\mathscr{M} = \mathscr{H}_n$ then $N = \pm M^*$.

The purpose of this paper is to determine the structure of the (p, q) numerical radius preservers for the other unknown cases. It has been shown in [16] that there is no simple characterization for (p, q) numerical range preservers if $(p, q) = (1, n)$ or $(2, n)$. One easily shows that the same conclusion holds for (p, q) numerical radius preservers. Therefore, these two cases are excluded from our discussion. For all other cases, the structure of the (p, q) numerical radius preservers is determined in the following two theorems.

THEOREM 1. *Let p, q, n be integers such that $1 < p < q \leqslant n$ but $(p, q) \neq (2, n)$. A linear operator ϕ on $\mathscr{H}_n$ satisfies*

$$r_{p,q}(\phi(A)) = r_{p,q}(A) \qquad \text{for all} \quad A \in \mathscr{H}_n$$

if and only if there exist a unitary matrix U and $\mu = \pm 1$ such that

$$\phi(A) = \mu UAU^* \qquad \text{for all} \quad A \in \mathscr{H}_n$$

or

$$\phi(A) = \mu U A^t U^* \qquad \text{for all} \quad A \in \mathscr{H}_n.$$

THEOREM 2. *Let p, q, n be integers such that $1 < p < q \leqslant n$ but $(p, q) \neq (2, n)$. A linear operator ϕ on $\mathbb{C}_{n \times n}$ satisfies*

$$r_{p,q}(\phi(A)) = r_{p,q}(A) \qquad \text{for all} \quad A \in \mathbb{C}_{n \times n}$$

if and only if there exist $\mu \in \mathbb{C}$ with $|\mu| = 1$, and a nonsingular matrix S, which is unitary if $q < n$, such that

$$\phi(A) = \mu S A S^{-1} \qquad \text{for all} \quad A \in \mathbb{C}_{n \times n}$$

or

$$\phi(A) = \mu S A^t S^{-1} \qquad \text{for all} \quad A \in \mathbb{C}_{n \times n}.$$

The sufficiency parts of the theorems can be verified readily. To establish the necessity parts, we shall show that a (p, q) numerical radius preserver ϕ on $\mathscr{M} = \mathscr{H}_n$ or $\mathbb{C}_{n \times n}$ is nonsingular and maps the set of rank one matrices into itself. By the results on rank one preservers (for example, see [5] and [15]) we conclude that

(i) if $\mathscr{M} = \mathscr{H}_n$ then (3) or (4) holds for some nonsingular $U \in \mathbb{C}_{n \times n}$ and $\mu = \pm 1$;

(ii) if $\mathscr{M} = \mathbb{C}_{n \times n}$ then (7) or (8) holds for some nonsingular $M, N \in \mathbb{C}_{n \times n}$.

The proof is then completed in Section 3 by showing that U, M, N etc. satisfy the conditions stated in the theorems.

We shall use $\{E_{11}, E_{12}, \dots, E_{nn}\}$ to denote the standard basis of $\mathbb{C}_{n \times n}$.

2. NONSINGULARITY AND RANK PRESERVING PROPERTIES

Suppose A has eigenvalues $\lambda_1, \dots, \lambda_n$. Let

$$h_{p,q}(A) = \max\left\{|E_p(\lambda_{i_1}, \dots, \lambda_{i_q})| : 1 \leqslant i_1 < \cdots < i_q \leqslant n\right\},$$

where $E_p(x_1, \dots, x_q)$ denotes the pth elementary symmetric function on $x_1, \dots, x_q$. Then (e.g., see [12, 17])

$$h_{p,q}(A) \leqslant r_{p,q}(A) \qquad \text{for all } A.$$

The equality holds if A is normal. Moreover, we have the following result (see [12, Theorem 7.1]).

LEMMA 2.1. *Let $1 \leqslant p \leqslant q < n$. A matrix $A \in \mathcal{M}$ satisfies $h_{p,q}(A) = 0$ if and only if A has less than p nonzero eigenvalues.*

LEMMA 2.2. *Let $1 < p < q \leqslant n$ but $(p, q) \neq (2, n)$. If ϕ is a (p, q) numerical radius preserver on $\mathcal{M}$, then ϕ is nonsingular.*

Proof. Suppose $q = n$. One can apply the arguments in the proof of [16, Lemma 3.1] to show that ϕ is nonsingular. (Notice that although the proof in [16] is for the complex case, the same argument can be used to deal with the hermitian case.)

Now suppose $q < n$. Assume that there exists a nonzero $A \in \mathcal{M}$ satisfying $\phi(A) = 0$. Then $r_{p,q}(A) = r_{p,q}(\phi(A)) = 0$. It follows that $h_{p,q}(A) = 0$. By Lemma 2.1, we have $k < p$, where k is the number of nonzero eigenvalues of A. Now suppose $k > 0$. Let U be a unitary matrix such that UAU^* is in upper triangular form (actually in diagonal form if $\mathcal{M} = \mathcal{H}_n$) with diagonal entries $\lambda_1, \dots, \lambda_n$ such that $|\lambda_1| \geqslant \cdots \geqslant |\lambda_n|$. If $B = U^*(\sum_{i=k+1}^{p} E_{ii})U$, then B is hermitian and $r_{p,q}(B) = 0$, as rank $B < p$. Since $A + B$ has p nonzero eigenvalues, namely, $\lambda_1, \dots, \lambda_k, 1, \dots, 1$, we have

$$\begin{aligned} 0 &< h_{p,q}(A + B) \leqslant r_{p,q}(A + B) \\ &= r_{p,q}(\phi(A + B)) = r_{p,q}(\phi(B)) = r_{p,q}(B) = 0, \end{aligned} \tag{9}$$

which is a contradiction.

Now if $\mathcal{M} = \mathbb{C}_{n \times n}$, it is possible that $A \neq 0$ and $k = 0$. Let $S \in \mathbb{C}_{n \times n}$ be such that SAS^{-1} is in upper triangular Jordan form with $(SAS^{-1})_{12} = 1$. If $B = S^{-1}(E_{21} + \sum_{i=3}^{p} E_{ii})S$, then $A + B$ has p nonzero eigenvalues, namely, $-1, 1, \dots, 1$, and hence (9) holds, which is a contradiction. ∎

To prove that ϕ maps the set of rank one matrices into itself, we need a characterization of rank one matrices in terms of the (p, q) numerical radius. The results for the hermitian case and the complex case are stated in Lemmas 2.3 and 2.4, respectively. We only give the proof for Lemma 2.4, which is the more difficult case. One readily sees that the argument also covers the hermitian case. It is worth mentioning that the results are motivated by Lemma 2 in [9].

LEMMA 2.3. *Let* $1 < p \leqslant q \leqslant n$ *but* $(p, q) \neq (2, n)$. *A nonzero matrix* $A \in \mathscr{H}_n$ *satisfies* rank $A = 1$ *if and only if for any* $B \in \mathscr{H}_n$ *there exists a positive number* ν *such that*

$$r_{p,q}(xA + B) \leqslant \nu(x + 1) \qquad \text{for all} \quad x > 0. \tag{10}$$

LEMMA 2.4. *Let* $1 < p < q \leqslant n$ *but* $(p, q) \neq (2, n)$, *and let* $A \in \mathbb{C}_{n\times n}$ *be nonzero.*

(a) *Suppose* $2 = p < q < n - 1$. *Then* rank $A = 1$ *if and only if* $r_{p,q}(A) = 0$.

(b) *Suppose* $2 < p < q \leqslant n$ *but* $(p, q) \neq (3, n)$. *Then* rank $A = 1$ *if and only if for any* $B \in \mathbb{C}_{n\times n}$ *there exists a positive number* ν *such that* (10) *holds.*

(c) *Suppose* $(p, q) = (2, n - 1), (3, n)$. *Then* rank $A = 1$ *or* $A^2 = 0$ *if and only if for any* $B \in \mathbb{C}_{n\times n}$ *there exists a positive number* ν *such that* (10) *holds.*

Proof. Notice that (a) is just a particular case of [12, Theorem 7.3(a)]. We prove parts (b) and (c) in the following.

Let $\mathscr{S}$ be the collection of all matrices C in $\mathbb{C}_{n\times n}$ satisfying the condition that for any $B \in \mathbb{C}_{n\times n}$ there exists a positive number ν such that

$$r_{p,q}(xC + B) \leqslant \nu(x + 1) \qquad \text{for all} \quad x > 0.$$

First we show that $A \in \mathscr{S}$ if rank $A = 1$. If rank $A = 1$, then there exists a nonsingular $S \in \mathbb{C}_{n\times n}$ such that

$$A' = SAS^{-1} = \begin{cases} E_{12} & \text{if all eigenvalues of } A \text{ equal zero,} \\ \lambda E_{11} & \text{if } A \text{ has a nonzero eigenvalue } \lambda. \end{cases} \tag{11}$$

For any $B \in \mathbb{C}_{n\times n}$ let $B' = SBS^{-1}$. Then

$$\sum_{r=0}^{q} t^r D_r^q(xA + B) = C_q(I + t(xA + B)) = C_q(S^{-1}[I + t(xA' + B')]S)$$

$$= C_q(S^{-1})C_q(I + t(xA' + B'))C_q(S),$$

and hence

$$D_p^q(xA+B) = C_q(S^{-1})D_p^q(xA'+B')C_q(S) \tag{12}$$

Because of (11), one easily checks that $C_q(I+t(xA'+B')) = xP(t)+Q(t)$, where $P(t)$ and $Q(t)$ are $\binom{n}{q}\times\binom{n}{q}$ matrices whose entries are polynomials of t. Thus

$$D_p^q(xA'+B') = xA''+B'' \qquad \text{with} \quad A'', B'' \in \mathbb{C}_{\binom{n}{q}\times\binom{n}{q}}.$$

Let $\tilde{A} = C_q(S)^{-1}A''C_q(S)$ and $\tilde{B} = C_q(S)^{-1}B''C_q(S)$. Then for any decomposable unit vector $u^\wedge \in \wedge^q(\mathbb{C}^n)$ and any $x>0$,

$$|\langle D_p^q(xA+B)u^\wedge, u^\wedge\rangle| \leqslant x|\langle \tilde{A}u^\wedge, u^\wedge\rangle| + |\langle \tilde{B}u^\wedge, u^\wedge\rangle| \leqslant \nu(x+1), \tag{13}$$

where $\nu = \max\{r(\tilde{A}), r(\tilde{B})\}$.

Next we show that if $(p, q) = (2, n-1)$ or $(3, n)$, and $A^2=0$, then $A\in\mathcal{S}$. Notice that if $A^2=0$, then A is a nilpotent matrix and all the Jordan blocks in its Jordan form have size less than 3. Assume rank $A=r$. Then there exists a nonsingular matrix S such that

$$A' = SAS^{-1} = \sum_{i=1}^{r} E_{i,r+i}. \tag{14}$$

For any $B\in\mathbb{C}_{n\times n}$, let $SBS^{-1}=B'$. Then the formula (12) is valid. If $(p, q) = (3, n)$, then $D_p^q(xA'+B') = C_3(xA'+B')$ is the sum of all the 3×3 principal minors of $xA'+B'$. By (14), one easily sees that $C_3(xA'+B') = x\alpha+\beta$ for some $\alpha, \beta\in\mathbb{C}$. Thus (13) holds with $\nu = \max\{|\alpha|, |\beta|\}$. If $(p, q) = (2, n-1)$, then every entry of $D_p^q(xA'+B')$ is obtained by taking the coefficient of t^2 in the determinant expansion of a certain $(n-1)\times(n-1)$ submatrix $Z(t, x)$ of $I+t(xA'+B')$. Notice that if $z(t,x) = \prod_{i=1}^{n-1} z_i(t, x)$, where $z_i(t, x)$'s are entries of $Z(t, x)$, is a summand of the determinant expansion of $Z(t, x)$ that has a nonzero contribution to the coefficient of t^2, then at least $n-3$ of the $z_i(t, x)$'s are chosen from the diagonal entries of

$I + t(xA' + B')$ and they are of the form $1 + tB'_{ii}$. By (14), one can check that at most one of the remaining two entries can be of the form $t(xA'_{ij} + B'_{ij})$, and hence $z(t, x) = \prod_{i=1}^{n} z_i(t, x) = xp(t) + q(t)$ for some polynomials $p(t)$ and $q(t)$. As a result,

$$D_p^q(xA' + B') = xA'' + B'' \qquad \text{with} \quad A'', B'' \in \mathbb{C}_{\binom{n}{q} \times \binom{n}{q}}.$$

Let $\tilde{A} = C_q(S)^{-1}A''C_q(S)$ and $\tilde{B} = C_q(S)^{-1}B''C_q(S)$. Then for any decomposable unit vector $u^{\wedge} \in \wedge^q(\mathbb{C}^n)$ and any $x > 0$, we see that (13) holds with $\nu = \max\{r(\tilde{A}), r(\tilde{B})\}$.

To complete the proof of (b) and (c), we show that (i) if $2 < p < q \leqslant n$, $(p, q) \neq (3, n)$, and rank $A > 1$, then $A \notin \mathcal{S}$, and (ii) if $(p, q) = (2, n - 1)$ or $(3, n)$ and if rank $A > 1$ and $A \in \mathcal{S}$, then $A^2 = 0$.

Suppose rank $A > 1$. Assume that A has k nonzero eigenvalues with $k > 0$. Suppose SAS^{-1} is in upper triangular Jordan form with $\lambda_i(SAS^{-1})_{ii} \neq 0$ for $i = 1, \ldots, k$, where $S \in \mathbb{C}_{n \times n}$ is nonsingular. Suppose $q < n$. If $k \geqslant p$, then $r_{p,q}(A) \geqslant h_{p,q}(A) > 0$ by Lemma 2.1. Let $B = 0$. We have $r_{p,q}(xA + B) = r_{p,q}(xA) = x^p r_{p,q}(A)$, and hence $A \notin \mathcal{S}$. Suppose $2 \leqslant k < p$. If $B = S^{-1}(\sum_{i=k+1}^{p} E_{ii})S$ then $r_{p,q}(xA + B) \geqslant h_{p,q}(xA + B) \geqslant x^k|\prod_{i=1}^{k} \lambda_k|$, and hence $A \notin \mathcal{S}$. Suppose $k = 1$. Since rank $A > 1$, we may assume that $(SAS^{-1})_{23} = 1$. Let B be such that

$$SBS^{-1} = \begin{cases} E_{32} + \sum_{i=4}^{p} E_{ii} & \text{if} \quad p \geqslant 3, \\ E_{32} & \text{if} \quad (p, q) = (2, n-1). \end{cases}$$

Then $xA + B$ has eigenvalues

$$x\lambda_1,\ x^{1/2},\ -x^{1/2},\ \overbrace{1, \ldots, 1}^{p-3}, 0, \ldots \qquad \text{if} \quad p \geqslant 3,$$

$$x\lambda_1,\ x^{1/2},\ -x^{1/2}, 0, \ldots \qquad \text{if} \quad (p, q) = (2, n-1).$$

As a result, for all $x > 0$

$$r_{p,q}(xA + B) \geqslant h_{p,q}(xA + B) \geqslant \begin{cases} |\lambda_1|x^2 & \text{if} \quad p \geqslant 3, \\ |\lambda_1|x^{3/2} & \text{if} \quad (p, q) = (2, n-1), \end{cases}$$

and hence $A \notin \mathcal{S}$. If $3 \leqslant p < q = n$, by [16, Lemma 3.4] and [1, Lemma 2.2] we conclude that $A \notin \mathcal{S}$.

Now suppose $k = 0$. If A has a Jordan block of size at least three. We may assume $(SAS^{-1})_{12} = (SAS^{-1})_{23} = 1$. Let B be such that

$$SBS^{-1} = \begin{cases} E_{31} + \sum_{i=4}^{p} E_{ii} & \text{if } \; p \geqslant 3, \\ E_{31} & \text{if } \; (p, q) = (2, n-1). \end{cases}$$

Then $xA + B$ has eigenvalues

$$\begin{aligned} &x^{2/3}, \eta x^{2/3}, \eta^2 x^{2/3}, \overbrace{1, \ldots, 1}^{p-3}, 0, \ldots && \text{if } \; p \geqslant 3, \\ &x^{2/3}, \eta x^{2/3}, \eta^2 x^{2/3}, 0, \ldots && \text{if } \; (p, q) = (2, n-1), \end{aligned}$$

where η is a primitive cube-root of unity. As a result, for all $x > 0$

$$r_{p,q}(xA + B) \geqslant h_{p,q}(xA + B) \geqslant \begin{cases} x^2 & \text{if } \; p \geqslant 3, \\ x^{4/3} & \text{if } \; (p, q) = (2, n-1), \end{cases}$$

and thus $A \notin \mathscr{S}$. Now suppose the sizes of all Jordan blocks of A are less than three. Since rank $A > 1$, we may assume $(SAS^{-1})_{12} = (SAS^{-1})_{34} = 1$. if $3 \leqslant p < q \leqslant n$ and $(p, q) \neq (3, n)$, let B be such that

$$SBS^{-1} = \begin{cases} E_{21} + E_{43} + \prod_{i=5}^{p} E_{ii} & \text{if } \; p \geqslant 4, \\ E_{21} + E_{43} & \text{if } \; 3 = p < q < n. \end{cases}$$

Then $xA + B$ has eigenvalues

$$\begin{aligned} &x^{1/2}, x^{1/2}, -x^{1/2}, -x^{1/2}, \overbrace{1, \ldots, 1}^{p-4}, 0, \ldots && \text{if } \; p \geqslant 4, \\ &x^{1/2}, x^{1/2}, -x^{1/2}, -x^{1/2}, 0, \ldots && \text{if } \; 3 = p < q < n. \end{aligned}$$

As a result, for all $x > 0$

$$r_{p,q}(xA + B) \geqslant h_{p,q}(xA + B) \geqslant \begin{cases} x^2 & \text{if } \; p \geqslant 4, \\ x^{3/2} & \text{if } \; 3 = p < q < n. \end{cases}$$

and hence $A \notin \mathscr{S}$. The proof is complete. ∎

Now we are ready to prove the rank preserving property of a (p, q) numerical radius preserver.

LEMMA 2.5. *Let* $1 < p < q \leqslant n$ *but* $(p, q) \neq (2, n)$. *If* ϕ *is a* (p, q) *numerical radius preserver on* $\mathscr{M} = \mathscr{H}_n$ *or* $\mathbb{C}_{n\times n}$, *then* ϕ *maps the set of rank one matrices into itself.*

Proof. Notice that if ϕ preserves the (p, q) numerical radius, then ϕ is nonsingular by Lemma 2.2. Suppose $\mathscr{M} = \mathscr{H}_n$. If rank $A = 1$, Lemma 2.3 will ensure that for any $B \in \mathscr{H}_n$ there exists $\nu > 0$ satisfying

$$r_{p,q}(x\phi(A) + B) = r_{p,q}(xA + \phi^{-1}(B)) \leqslant \nu(x+1) \qquad \text{for all} \quad x > 0.$$

It follows from Lemma 2.3 again that rank $\phi(A) = 1$.

Suppose $\mathscr{M} = \mathbb{C}_{n\times n}$. Suppose $2 = p < q < n - 1$. If rank $A = 1$, then by Lemma 2.4(a)

$$r_{p,q}(\phi(A)) = r_{p,q}(A) = 0.$$

Since $\phi(A)$ is nonzero by Lemma 2.2, it follows that from Lemma 2.4(a) that rank $\phi(A) = 1$. Suppose $2 < p < q \leqslant n$ but $(p, q) \neq (3, n)$. If rank $A = 1$, one can use Lemma 2.4(b) and the same argument in the hermitian case to conclude that rank $\phi(A) = 1$.

Now suppose $\mathscr{M} = \mathbb{C}_{n\times n}$, and $(p, q) = (2, n-1)$ or $(3, n)$. We claim that if $A \in \mathscr{S}$, where $\mathscr{S}$ is the set of rank one matrices in $\mathscr{M}$ with one nonzero eigenvalue, then rank $\phi(A) = 1$. Notice that if rank $B = 1$, then B belongs to the closure of $\mathscr{S}$. As a result, $\phi(B)$ belongs to the closure of the set of rank one matrices, and hence rank $\phi(B) \leqslant 1$. Since ϕ is nonsingular by Lemma 2.2, we see than rank $\phi(B) = 1$.

If our claim is not true, then there exists $A = \alpha S E_{11} S^{-1}$, where $S \in \mathbb{C}_{n\times n}$ is nonsingular, such that $B = \phi(A)$ has rank larger than 1. Since for any $C \in \mathbb{C}_{n\times n}$ Lemma (2.4(c) ensures that there exists $\nu > 0$ satisfying

$$r_{p,q}(xB + C) = r_{p,q}(xA + \phi^{-1}(C)) \leqslant \nu(x+1) \qquad \text{for all} \quad x > 0,$$

we conclude that $B^2 = 0$. Thus $1 < \text{rank } B = r \leqslant n/2$, and there exists a nonsingular $T \in \mathbb{C}_{n\times n}$ such that

$$B = T(E_{1s} + E_{2,s+1} + \cdots + E_{r,s+r-1})T^{-1},$$

where

$$x = \begin{cases} (n+2)/2 & \text{if } n \text{ is even,} \\ (n+3)/2 & \text{if } n \text{ is odd.} \end{cases}$$

Let $\mathscr{Y}$ be the subspace of $\mathbb{C}_{n\times n}$ spanned by the matrices $TE_{ij}T^{-1}$, $1 \leqslant i < s \leqslant j \leqslant n$. By Lemma 2.4(c), for any $R \in \mathscr{Y}$, $C \in \mathbb{C}_{n\times n}$, and $\lambda \in \mathbb{C}$, there exists $\nu > 0$ such that

$$r_{p,q}\left(x\left[\lambda A + \phi^{-1}(R)\right] + C\right) = r_{p,q}\left(x(\lambda B + R) + \phi(C)\right) \leqslant \nu(x+1)$$

$$\text{for all} \quad x > 0.$$

Since $A = \alpha S E_{11} S^{-1}$, the matrix $\lambda A + \phi^{-1}(R)$ is not nilpotent for sufficiently large $\lambda > 0$. By Lemma 2.4(c), $\operatorname{rank}[\lambda A + \phi^{-1}(R)] = 1$ for sufficiently large $\lambda > 0$. Hence $S^{-1}\phi^{-1}(R)S$ is either of the form $\sum_{i=1}^{n} \beta_i E_{i1}$ or of the form $\sum_{i=1}^{n} \beta_i E_{1i}$. Since this is true for arbitrary $R \in \mathscr{Y}$, we see that $\phi^{-1}(\mathscr{Y}) \subset \mathscr{C}$ or $\mathscr{R}$, where $\mathscr{C}$ ($\mathscr{R}$) is spanned by the matrices $SE_{i1}S^{-1}$ ($SE_{1i}S^{-1}$), $1 \leqslant i \leqslant n$. Since $\dim \mathscr{Y} = n^2/4$ if n is even, $\dim \mathscr{Y} = (n^2-1)/4$ if n is odd, and $\dim \mathscr{C} = \dim \mathscr{R} = n$, we see that $n \leqslant 4$. Since $(p,q) = (2, n-1)$ or $(3, n)$, we conclude that $n = 4$ and hence $B = \phi(A) = T(E_{13} + E_{24})T^{-1}$. Since ϕ is nonsingular, by a dimension argument we see that (i) $\phi^{-1}(\mathscr{Y}) = \mathscr{C}$ or (ii) $\phi^{-1}(\mathscr{Y}) = \mathscr{R}$. If (i) holds, we shall show that there exists a matrix C which is a nontrivial linear combination of $SE_{12}S^{-1}$ and $SE_{13}S^{-1}$ such that $\phi(C) \in \mathscr{Y}$, which will be a contradiction. One can show that (ii) cannot hold by a similar argument, and the proof of the lemma will then be completed.

Let $R = SE_{12}S^{-1}$, and let

$$\phi(R) = T\begin{bmatrix} K & L \\ M & N \end{bmatrix}T^{-1}, \qquad \text{where} \quad K, L, M, N \in \mathbb{C}_{2\times 2}.$$

By Lemma 2.4(c), for any $C \in \mathbb{C}_{4\times 4}$ and $\lambda \in \mathbb{C}$, there exists $\nu > 0$ such that

$$r_{p,q}\left(x\left[\lambda B + \phi(R)\right] + C\right) = r_{p,q}\left(x(\lambda A + R) + \phi^{-1}(C)\right) \leqslant \nu(x+1)$$

$$\text{for all} \quad x > 0.$$

By Lemma 2.4(c), if $Y(\lambda) = \lambda B + \phi(R)$, then

(i) rank $Y(\lambda)) = 1$ or
(ii) rank $Y(\lambda)) > 1$ and

$$T^{-1}\big(y(\lambda)^2\big)T = \lambda\begin{bmatrix} M & K+N \\ 0 & M \end{bmatrix} + \begin{bmatrix} K^2+LM & KL+LN \\ MK+NM & ML+N^2 \end{bmatrix} = 0.$$

Since condition (ii) holds for all sufficiently large $\lambda > 0$, we conclude that $M = K + N = K^2 = N^2 = KL + LN = 0$. Applying the same argument to $R(\mu) = S(E_{12} + \mu E_{13})S^{-1}$, $\mu \in \mathbb{C}$, one sees that

$$\phi(R(\mu)) = T\begin{bmatrix} K_\mu & L_\mu \\ M_\mu & N_\mu \end{bmatrix}T^{-1},$$

$M_\mu = K_\mu + N_\mu = K_\mu^2 = N_\mu^2 = K_\mu L_\mu + L_\mu N_\mu = 0$. Since $K_\mu = K_0 + \mu(K_1 - K_0)$ is a nilpotent matrix for any $\mu \in \mathbb{C}$, one readily sees that either K_0 is a zero matrix or K_1 is a multiple of $K_0 = K$. Thus there exists a nontrivial combination of $\phi(SE_{12}S^{-1})$ and $\phi(SE_{13}S^{-1})$ having the form

$$T\begin{bmatrix} 0 & L' \\ 0 & 0 \end{bmatrix}T^{-1} \in \mathscr{V},$$

which is the desired contradiction. ∎

3. PROOF OF THE THEOREMS

Proof of Theorem 1. Let $1 < p < q \leqslant n$ and $(p, q) \neq (2, n)$. Suppose ϕ is a (p, q) numerical radius preserver on $\mathscr{H}_n$. By Lemmas 2.2 and 2.5, ϕ is nonsingular and maps the set of rank one matrices into itself. By [5, Lemma 2], there exists $\mu = \pm 1$ such that $\mu\phi$ is of the form

(i) $A \mapsto SAS^*$ for all A or
(ii) $A \mapsto SA^tS^*$ for all A.

To complete the proof we need to show that S is unitary. Now suppose (i) holds and S has singular value decomposition UDV^* where U, V are unitary and $D = \operatorname{diag}(d_1, \dots, d_n)$ with $d_1 \geqslant \cdots \geqslant d_n \geqslant 0$. Let $A_1 = V(I_p \oplus 0_{n-p})V^*$ and $A_2 = V(0_{n-p} \oplus I_p)V^*$. Since $r_{p,q}(\phi(A_i)) = r_{p,q}(A_i) = h_{p,q}(A_i) = 1$ for $i = 1, 2$, in view of the discussion before Lemma 2.1 it follows that $d_1 = \cdots = d_n$. Hence S is unitary. If (ii) holds, we can get the conclusion by a similar argument. ∎

To prove Theorem 2 we need three more lemmas.

LEMMA 3.1. *Let $1 < p < q \leqslant n$, and let $A \in \mathbb{C}_{n\times n}$.*

(a) *If* rank $A = p$, *then* $r_{p,q}(A) \leqslant r_p^{\wedge}(A)$.

(b) *If A is unitarily similar to $A_1 \oplus A_2$, where $A_1 \in \mathbb{C}_{(p-1)\times(p-1)}$ is nonsingular and* rank $A_2 = 1$, *then*

$$r_{p,q}(A) = r_p^{\wedge}(A) = |\det A_1| r(A_2).$$

Proof. (a): Suppose $X \in \mathbb{C}_{n\times q}$ satisfies $X^*X = I_q$ and $|E_p(X^*AX)| = r_{p,q}(A)$. Since X^*AX has at most p nonzero eigenvalues, we have

$$|E_p(X^*AX)| \leqslant h_{p,p}(X^*AX) \leqslant r_p^{\wedge}(X^*AX) \leqslant r_p^{\wedge}(A).$$

(b): We may assume that $A = A_1 \oplus A_2$ with A_1 and A_2 satisfying the hypotheses of the lemma. Furthermore, we may assume that $A_2 = \alpha E_{11} + \beta E_{12}$, where $E_{11}, E_{12} \in \mathbb{C}_{(n-p+1)\times(n-p+1)}$. Then

$$C_p(A) = (\det A_1)(\alpha E_{11} + \beta E_{12}), \qquad \text{where} \quad E_{11}, E_{12} \in \mathbb{C}_{\binom{n}{p}\times\binom{n}{p}}.$$

Thus $r_p^{\wedge}(A) \leqslant r(C_p(A)) = |\det A_1| r(\alpha E_{11} + \beta E_{12}) = |\det A_1| r(A_2)$.

Let U be a unitary matrix such that $U^*AU = A_1 \oplus A_2' \oplus 0_{n-p-1}$, where $A_2' \in \mathbb{C}_{2\times 2}$ and the modulus of the (1, 1) entry of A_2' equals $r(A_2)$. Let $X \in \mathbb{C}_{n\times q}$ be obtained from I_n by removing the $(p+1)$th column and the last $n-q-1$ columns. Then $X^*U^*AUX = A_1 \oplus r(A_2)I_1 \oplus 0_{q-p}$ and so

$$r_p^{\wedge}(A) \leqslant |\det A_1| r(A_2) = |E_p(X^*U^*AUX)| \leqslant r_{p,q}(A).$$

Combining this with the result in (a), we get the conclusion. ■

LEMMA 3.2. *Suppose $n \geqslant 2$ and $A \in \mathbb{C}_{n\times n}$ satisfies* rank $A = 1$. *Then*

$$r(A) = \frac{|\mathrm{tr}\, A| + \sqrt{\mathrm{tr}(A^*A)}}{2}.$$

Proof. Notice that if rank $A = 1$, then A is unitarily similar to

$$\begin{bmatrix} \alpha & \beta \\ 0 & 0 \end{bmatrix} \oplus O_{n-2},$$

where $\alpha = \operatorname{tr} A$ and $|\beta| = \sqrt{\operatorname{tr}(A^*A) - |\operatorname{tr} A|^2}$. It follows (for example, see [4, 18]) that $W(A)$ is an elliptical disk with 0 and $\operatorname{tr} A$ as foci, and with $\sqrt{|\beta|^2 + |\alpha|^2} = \sqrt{\operatorname{tr}(A^*A)}$ as the major axis. Notice that the elliptical disk may degenerate to a line segment with endpoints 0 and $\operatorname{tr} A$ if $\beta = 0$. In any event, $r(A) = \{|\operatorname{tr} A| + \sqrt{\operatorname{tr}(A^*A)}\}/2$, as asserted. ■

LEMMA 3.3. *Let $n \geqslant 2$ and $D, N \in \mathbb{C}_{n\times n}$, where N is nonsingular and $D = \operatorname{diag}(d_1, \dots, d_n)$ with $d_1 \geqslant \cdots \geqslant d_n > 0$. Suppose $r(DAN) = r(DUAU^*N)$ for any unitary matrix U and rank one matrix A. Then either*

(i) *N is a diagonal matrix or*
(ii) *$n = 2$, $d_1 = d_2$, and*

$$N = \begin{bmatrix} 0 & y \\ -y & 0 \end{bmatrix} \qquad \text{for some nonzero } y \in \mathbb{C}.$$

Proof. Let D and N satisfy the hypotheses of the lemma. Let

$$A = \begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix} \quad \text{and} \quad A' = \begin{bmatrix} a & 0 \\ b' & 0 \end{bmatrix},$$

where $a \in \mathbb{C}$, $b, b' \in \mathbb{C}^{n-1}$ with $\|b\| = \|b'\|$. Then $A' = UAU^*$ for some unitary U, and therefore $r(DAN) = r(DA'N)$ by the hypotheses of the lemma. Suppose the first row of N equals $[x, y^*]$ where $x \in \mathbb{C}$ and $y \in \mathbb{C}^{n-1}$. By Lemma 3.2, if $D' = \operatorname{diag}(d_2, \dots, d_n)$, then

$$|xd_1a + y^*D'b| + \gamma = 2r(DAN) = 2r(DA'N) = |xd_1a + y^*D'b'| + \gamma'. \tag{15}$$

where

$$\gamma = \sqrt{(|x|^2 + \|y\|^2)(|d_1a|^2 + \|D'b\|^2)},$$

$$\gamma' = \sqrt{(|x|^2 + \|y\|^2)(|d_1a|^2 + \|D'b'\|^2)}.$$

Suppose $n \geqslant 3$. If $x \neq 0$, choose a, b, b' such that $a \neq 0$, $xd_1 a$ is not purely imaginary, and $b = -b' = y$. Then (15) implies that $y = 0$. If $x = 0$, then $y \neq 0$ and one can choose $b, b' \in \mathbb{C}^{n-1}$ such that $|y^* D'b| = 0 < |y^* D'b'| = k_0$. Let $k_1 = \|y\|^2 |d_1|^2 \neq 0$, $k_2 = \|y\|^2 \|D'b\|^2$, and $k_3 = \|y\|^2 \|D'b'\|^2$. Then (15) implies that

$$\sqrt{k_1 |a|^2 + k_2} = k_0 + \sqrt{k_1 |a|^2 + k_3}$$

for all $a \in \mathbb{C}$, which is impossible. By the same arguments, one can show that only the diagonal entry in each row can be nonzero. Thus N is a diagonal matrix.

Suppose $n = 2$. Then $\gamma = \gamma'$ for any $a, b, b' \in \mathbb{C}$ with $|b| = |b'|$. It follows from Equation (15) that $y = 0$ or $x = 0$. By the same arguments, one can show that there can only be one nonzero element in the second row. Thus either

(i) N is a diagonal matrix, or

(ii) $N = \begin{bmatrix} 0 & y \\ z & 0 \end{bmatrix}$ for some nonzero $y, z \in \mathbb{C}$.

Suppose (ii) holds. Let

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad A' = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then $r(DAN) = r(DA'N)$ implies $d_1|y| = d_2|z|$. Let

$$A = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}, \quad A' = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}, \quad \text{and} \quad A'' = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}.$$

Then $r(DAN) = r(DA'N)$ implies $d_1|y| = d_2|y|$. As a result, $d_1 = d_2$ and $|y| = |z|$. Also, we have $r(DA''N) = r(DAN)$, and hence $z = -y$. ■

Proof of Theorem 2. Let $1 < p < q \leqslant n$ and $(p, q) \neq (2, n)$. Suppose ϕ is a (p, q) numerical radius preserver on $\mathbb{C}_{n \times n}$. By Lemma 2.5, ϕ maps the set of rank one matrices into itself. By [15], there exist nonsingular M, $N \in \mathbb{C}_{n \times n}$ such that ϕ is of the form (i) $A \mapsto MAN$ for all A, or (ii) $A \mapsto MA^t N$ for all A. If $q = n$, then condition (i) implies

$$r_{p,n}(A) = r_{p,n}(MAN) = r_{p,n}(A(NM)) \qquad \text{for all } A.$$

By [12, Theorem 6.3], we see that $NM = \mu I$ with $|\mu| = 1$. Thus the conclusion of Theorem 2 holds. By similar arguments, one can show that the conclusion of Theorem 2 is valid if (ii) holds.

Now suppose $1 < p < q < n$. We may assume condition (i) holds; otherwise we can consider ψ defined by $\psi(A) = \phi(A^t)$ instead of ϕ. Let M have singular value decomposition UDV, where U and V are unitary, and $D = \operatorname{diag}(\mu_1, \dots, \mu_n)$ with $\mu_1 \geqslant \cdots \geqslant \mu_n > 0$. We may assume that $U = V = I$; otherwise we can consider ψ defined by $\psi(A) = U^*\phi(V^*AV)U$ instead of ϕ. Let

$$N = \begin{bmatrix} N_1 & N_2 \\ N_3 & N_4 \end{bmatrix} \qquad \text{with} \quad N_4 \in \mathbb{C}_{p\times p},$$

and let $\eta_1 \geqslant \cdots \geqslant \eta_n > 0$ be the singular values of N. We claim that $N_2 = 0 = N_3^t$. Suppose $A = 0_{n-p} \oplus I_p$. Then $\phi(A) = D'N$, where $D' = \operatorname{diag}(0, \dots, 0, \mu_{n-p+1}, \dots, \mu_n)$, has rank p. By Lemma 3.1(a), we have

$$\begin{aligned} 1 = r_{p,q}(A) &= r_{p,q}(\phi(A)) \leqslant r_p^{\wedge}(\phi(A)) \\ &\leqslant \|C_p(\phi(A))\| \leqslant \|C_p(D')\|\,\|C_p(N)\| = \prod_{i=1}^{p}(\mu_{n-p+i}\eta_i), \end{aligned} \tag{16}$$

where $\|\cdot\|$ denotes the spectral norm. Now suppose X and Y are unitary matrices such that $XN^{-1}Y = \operatorname{diag}(\eta_1^{-1}, \dots, \eta_n^{-1})$. Let $\tilde{A} = X^*(I_p \oplus 0_{n-p})X$. Then

$$\begin{aligned} 1 &= r_{p,q}(\tilde{A}) = r_{p,q}\big(\phi^{-1}(\tilde{A})\big) \leqslant r_p^{\wedge}\big(\phi^{-1}(\tilde{A})\big) \\ &\leqslant \|C_p(\phi^{-1}(A))\| = \|C_p(D^{-1}\tilde{A}N^{-1})\| = \|C_p(D^{-1}\tilde{A}N^{-1}Y)\| \\ &\leqslant \|C_p(D^{-1}X^*)\|\,\|C_p\big((I_p \oplus 0_{n-p})XN^{-1}Y\big)\| = \prod_{i=1}^{n}(\mu_{n-p+i}\eta_i)^{-1}. \end{aligned}$$

We conclude that $1 = \prod_{i=1}^{p}(\mu_{n-p+i}\eta_i)$, and hence all the inequalities in (16) are equalities. Since $\phi(A)$ has rank p and $r_p^{\wedge}(\phi(A) = \|C_p(\phi(A))\|$, by [6, Theorem 5] there exists a unitary matrix X such that $X\phi(A)X^* = 0_{n-p} \oplus B$

with $|\det B| = \|C_p(\phi(A))\|$. Since $\phi(A)$ is of the form

$$\begin{bmatrix} 0 & 0 \\ A_3 & A_4 \end{bmatrix},$$

we conclude that $X = X_1 \oplus X_2$ with $X_2 \in \mathbb{C}_{p\times p}$, $A_3 = 0$, and hence $N_3 = 0$. Furthermore, since $|\det B| = |\det A_4|$, we have $\prod_{i=1}^{p} \eta_i = |\det N_4|$. It follows that $\prod_{i=1}^{p} \eta_i = |\det N_4| \leqslant r_p^{\wedge}(N) \leqslant \prod_{i=1}^{p} \eta_i$, and hence (for example, see [6, Lemma 2]) $N_2 = 0$. Therefore our claim is proved.

Next we show that either

(i) N_1 is a diagonal matrix or

(ii) $n - p = 2$, $\mu_1 = \mu_2$ and $N_1 = \begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix}$.

Let $D = D_1 \oplus D_2$ with $D_2 \in \mathbb{C}_{p\times p}$. Suppose $A = A_1 \oplus A_2$, where A_1 is a rank one matrix and $A_2 = (I_{p-1} \oplus 0_1)N_2^{-1}$. Then for any $(n-p)\times(n-p)$ unitary matrix U,

$$r_{p,q}(D_1 A_1 N_1 \oplus D_2 A_2 N_2) = r_{p,q}(DAN) = r_{p,q}(A) = r_{p,q}(UA_1U^* \oplus A_2)$$
$$= r_{p,q}(D_1 UA_1 U^* N_1 \oplus D_2 A_2 N_2).$$

By Lemma 3.1(b),

$$r(D_1 A_1 N_1) = r(D_1 UA_1 U^* N_1).$$

By Lemma 3.3, we get the desired conclusion on N_1.

Now apply the arguments in the preceding paragraphs to $\phi^{-1}(A) = D^{-1}AN^{-1}$; we see that N can also be written as $N_1' \oplus N_2'$ with $N_1' \in \mathbb{C}_{p\times p}$ such that either

(i) N_2' is a diagonal matrix, or

(ii) $n - p = 2$, $\mu_{n-1} = \mu_n$, and $N_2' = \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix}$.

If both N_1 and N_2' are diagonal matrices, then N can be written as $[x] \oplus L \oplus [y]$ with $L \in \mathbb{C}_{(n-2)\times(n-2)}$. Let $A = E_{11} + (\sum_{i=2}^{p} \mu_i^{-1} E_{ii})N^{-1}$ and $A' = E_{nn} + (\sum_{i=2}^{p} \mu_i^{-1} E_{ii})N^{-1}$. Then A and A' are unitarily similar, and by Lemma 3.1(b),

$$\mu_1|x| = r_{p,q}(DAN) = r_{p,q}(A) = r_{p,q}(A') = r_{p,q}(DA'N) = \mu_n|y|.$$

Let $A = 2E_{1n} + (\sum_{i=2}^{p} \mu_i^{-1}E_{ii})N^{-1}$ and $A' = 2E_{n1} + (\sum_{i=2}^{p} \mu_i^{-1}E_{ii})N^{-1}$. Then A and A' are unitarily similar, and by Lemma 3.1(b),

$$\mu_1|y| = r_{p,q}(DAN) = r_{p,q}(A) = r_{p,q}(A') = r_{p,q}(DA'N) = \mu_n|x|.$$

It follows that $\mu_1 = \mu_n$ and hence

$$r_{p,q}(A) = r_{p,q}(DAN) = r_{p,q}(A(\mu_i N)) \qquad \text{for all } A. \tag{17}$$

By [12, Theorem 6.3], we see that $\mu_1 N = \lambda I$ with $|\lambda| = 1$. Thus the conclusion of Theorem 2 follows.

Suppose at least one of N_1 or N_2' is not a diagonal matrix. Then $n = q + 1 = p + 2$.

First, assume that N_1 is a diagonal matrix but N_2' is not. Then $\mu_{n-1} = \mu_n$ and N is of the form

$$[x] \oplus L \oplus \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix} \qquad \text{with} \quad L \in \mathbb{C}_{(p-1)\times(p-1)}.$$

Let $A = 2E_{11} + (\sum_{i=2}^{p} \mu_i^{-1}E_{ii})N^{-1}$ and $A' = 2E_{nn} + (\sum_{i=2}^{p} \mu_i^{-1}E_{ii})N^{-1}$. Then A and A' are unitarily similar, and by Lemma 3.1(b),

$$2\mu_1|x| = r_{p,q}(DAN) = r_{p,q}(A) = r_{p,q}(A') = r_{p,q}(DA'N) = \mu_n|b|. \tag{18}$$

Let $A = 2E_{n1} + (\sum_{i=2}^{p} \mu_i^{-1}E_{ii})N^{-1}$ and $A' = 2E_{n,n-1} + (\sum_{i=2}^{p} \mu_i^{-1}E_{ii})N^{-1}$. Then A and A' are unitarily similar, and by Lemma 3.1(b),

$$\mu_n|x| = r_{p,q}(DAN) = r_{p,q}(A) = r_{p,q}(A') = r_{p,q}(DA'N) = 2\mu_n|b|.$$

Combining with (18), we have $4\mu_1 = \mu_n$, contradicting the fact that $\mu_1 \geqslant \mu_n$.

Next assume that N_2' is a diagonal matrix but N_1 is not. Then $\mu_1 = \mu_2$ and N is of the form

$$\begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix} \oplus L \oplus [y] \qquad \text{with} \quad L \in \mathbb{C}_{(p-1)\times(p-1)}.$$

Let $A = 2E_{21} + (\sum_{i=3}^{p+1} \mu_i^{-1}E_{ii})N^{-1}$ and $A' = 2E_{n1} + (\sum_{i=3}^{p+1} \mu_i^{-1}E_{ii})N^{-1}$. Then A and A' are unitarily similar, and by Lemma 3.1(b),

$$2\mu_2|a| = r_{p,q}(DAN) = r_{p,q}(A) = r_{p,q}(A') = r_{p,q}(DA'N) = \mu_n|a|,$$

contradicting the fact that $\mu_2 \geqslant \mu_n$.

Finally, assume that both N_1 and N_2' are not diagonal. Then $\mu_1 = \mu_2$, $\mu_{n-1} = \mu_n$, and N is of the form

$$\begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix} \oplus L \oplus \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix} \quad \text{with} \quad L \in \mathbb{C}_{(p-2)\times(p-2)}.$$

Let $A = E_{11} + E_{22} + (\Sigma_{i=3}^{p} \mu_i^{-1} E_{ii})N^{-1}$ and $A' = E_{nn} + E_{n-1,n-1} + (\Sigma_{i=3}^{p} \mu_i^{-1} E_{ii})N^{-1}$. Then A and A' are unitarily similar and

$$(\mu_1|a|)^2 = r_{p,q}(DAN) = r_{p,q}(A) = r_{p,q}(A') = r_{p,q}(DA'N) = (\mu_n|b|)^2. \tag{19}$$

Let $A = E_{12} + E_{n1} + (\Sigma_{i=3}^{p} \mu_i^{-1} E_{ii})N^{-1}$ and $A' = E_{1n} + E_{n,n-1} + (\Sigma_{i=3}^{p} \mu_i^{-1} E_{ii})N^{-1}$. Then A and A' are unitarily similar, and by Lemma 3.1(b),

$$\mu_1 \mu_n |a|^2/2 = r_{p,q}(DAN) = r_{p,q}(A) = r_{p,q}(A')$$

$$= r_{p,q}(DA'N) = \mu_1 \mu_n |b|^2/2.$$

Combining with (19), we have $\mu_1 = \mu_n$ and hence (17) holds. By [12, Theorem 6.3], we see that $\mu_1 N = \lambda I$ with $|\lambda| = 1$. This contradicts the fact that N_1 and N_2' are not diagonal. ∎

4. REMARKS

By Theorems 1 and 2 and the results in [7, 9–11], we see that ϕ is a (p, q) numerical radius preserver if and only if $\mu\phi$ is a (p, q) numerical range preserver for some scalar μ with $|\mu| = 1$. It would be nice to find an independent proof for this fact. Then one could obtain the characterization of (p, q) numerical radius preservers via (p, q) numerical range preservers. In fact, our proof is computational and is rather long; it would be nice to have a shorter conceptual proof for our results.

There are many other generalizations of numerical range and numerical radius. It would be nice to have a proof or a counterexample for the following statement:

A linear operator ϕ preserves a certain kind of generalized numerical radius if and only if $\mu\phi$ preserves the corresponding type of numerical range for some scalar μ with $|\mu| = 1$.

More generally, one can regard $W_{p,q}(\cdot)$ and $r_{p,q}(\cdot)$ as special cases of *unitary similarity invariant* functions on matrices, i.e., those functions F satisfying $F(A) = F(UAU^*)$ for all A and for all unitary matrix U. It would be interesting to characterize the linear preservers of other unitary similarity invariant functions. Another interesting problem is to characterize those unitary similarity invariant functions F such that a linear operator ϕ preserves F if and only if (3) or (4) holds for some unitary matrix U and $\mu = 1$.

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REFERENCES

1 L. R. Beasley, Linear transformations on matrices: The invariance of the third elementary symmetric function, *Canad. J. Math.* 22:746–752 (1970).
2 N. Bebiano and C. K. Li, A brief survey on decomposable numerical ranges, *Linear and Multilinear Algebra*, 32:179–190 (1992).
3 N. Bebiano, C. K. Li, and J. da Providência, Some results on the numerical range of a derivation, *SIAM J. Matrix Anal. Appl.*, 14:1084–1095 (1993).
4 R. A. Horn and C. R. Johnson, *Topics in Matrix Analysis*, Cambridge U.P., 1991.
5 C. R. Johnson and S. Pierce, Linear maps on Hermitian matrices: The stabilizer of an inertia class II, *Linear and Multilinear Algebra* 19:21–31 (1986).
6 C. K. Li, Decomposable numerical radius and numerical radius of a compound matrix, *Linear Algebra Appl.* 76:45–58 (1986).
7 C. K. Li, Linear operators preserving the numerical radius of matrices, *Proc. Amer. Math. Soc.* 99:601–608 (1987).
8 C. K. Li, Linear operators preserving the higher numerical radius of matrices, *Linear and Multilinear Algebra* 21:63–73 (1987).
9 C. K. Li, B. S. Tam, and N. K. Tsing, Linear operators preserving the (p, q) numerical range, *Linear Algebra Appl.* 110:75–89 (1988).
10 C. K. Li and N. K. Tsing, Linear operators preserving the decomposable numerical radius, *Linear and Multilinear Algebra* 23:333–341 (1988).
11 C. K. Li and N. K. Tsing, Duality between some linear preserver problems: The invariance of the c-numerical range, the c-numerical radius and certain matrix sets, *Linear and Multilinear Algebra* 23:353–362 (1988).

12 C. K. Li and N. K. Tsing, The numerical range of derivatives, *Linear Algebra Appl.* 119:97–119 (1989).
13 M. Marcus, *Finite Dimensional Multilinear Algebra*, parts I, II, Marcel Dekker, New York, 1973, 1975.
14 M. Marcus, Derivations, Plücker relations, and the numerical range, *Indiana Univ. Math. J.* 22:1137–1149 (1973).
15 M. Marcus and B. Moyls, Linear transformations on algebras of matrices, *Canad. J. Math.* 11:61–66 (1959).
16 M. Marcus and R. Purves, Linear transformations on algebras of matrices: The invariance of the elementary symmetric functions, *Canad. J. Math.* 11:383–396 (1959).
17 M. Marcus and M. Sandy, Vertex points in the numerical range of a derivation, *Linear and Multilinear Algebra* 21:385–394 (1987).
18 F. D. Murnaghan, On the field of values of a square matrix, *Proc. Nat. Acad. Sci. U.S.A.* 18:246–248 (1932).

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